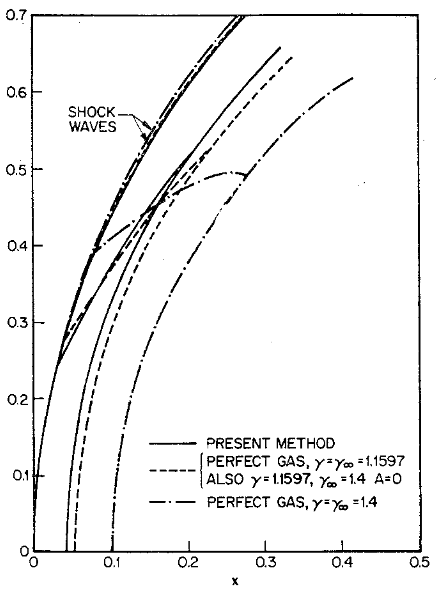


**Fig. 3 Comparison of body shapes and sonic lines; $M = 25$, $z = 200{,}000$ ft.**

$A = 0$), and for a specific-heat ratio in the shock layer different from the freestream value, again with $A = 0$. Further note the large difference between these coincident solutions and the solution for a perfect gas having a ratio of specific heats equal to the freestream value, and the difference between the coincident solutions and that of the present method. From these comparisons one may conclude that the main parameters affecting the solution are the ratio of specific heats in the shock layer and the constant $A$, whereas the value of the freestream specific-heat ratio has relatively little affect. Considering now the perfect-gas solution having $\gamma = 1.1597$ and the solution by the present method, one notes that, even though the shock waves supported by a sphere in both cases are coincident, the body and sonic-line positions are significantly different. Hence, large errors can be incurred in approximating a real-gas flow by a perfect gas having a constant ratio of specific heats corresponding to an average value of the actual flow in the shock layer. Even larger error is incurred by assuming the specific-heat ratio to be constant at the freestream value. However, the error is seen to be small in approximating a real-gas flow by one having a constant ratio of specific heats different from that of freestream, along with an appropriate value of the constant $A$ as determined by Eq. (1) and the relation $A = \tilde{A}/q_\infty^2$.

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# Heat Transfer to a Sphere for Free Molecule Flow of a Nonuniform Gas

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**Theoretical relations are obtained for the heat transfer to a sphere for the free molecule flow of a nonuniform gas. The sphere diameter is assumed to be small compared to the molecular mean free path, and the gas flow is assumed to have arbitrary viscous stress and heat flux terms present, such as the case encountered in a boundary layer. The effect of these nonuniformities on the heat transfer to the sphere and its equilibrium temperature are determined theoretically.**

### Introduction

THE purpose of this analysis is to obtain theoretical relations for the heat transfer to spheres in a free molecule flow of a nonuniform gas. A nonuniform gas is defined as one whose distribution function deviates from the Maxwellian equilibrium distribution because of the presence of viscous stresses and heat flux terms. This problem arose in connection with free molecule spherical probes used for surveying boundary layers in supersonic low-density nozzles[2] as a direct extension to similar University of California free molecule cylindrical probes.[1] The method of approach closely parallels the analysis used by Bell and Schaaf[1] in the case of cylinders.

### Procedure

The energy balance for a differential area $dA$ in the absence of radiation is written as

$$dQ = dE_i - dE_r \tag{1}$$

where $dE_i$ is the incident energy flux, $dE_r$ the re-emitted energy flux, and $dQ$ the net heat loss per unit time from the surface element. The thermal accommodation coefficient is defined as

$$\alpha = (dE_i - dE_r)/(dE_i - dE_w) \tag{2}$$

where $dE_w$ is the energy flux that would be re-emitted from the surface if all molecules were re-emitted with a Maxwellian distribution corresponding to the surface temperature $T_w$. Introducing this definition in Eq. (1), one gets

$$(1/\alpha)dQ = dE_i - dE_w \tag{3}$$

The incident energy flux per unit area is

$$\frac{dE_i}{dA} = \frac{m}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} (u'^2 + v'^2 + w'^2)u'f du'dv'dw' \tag{4}$$

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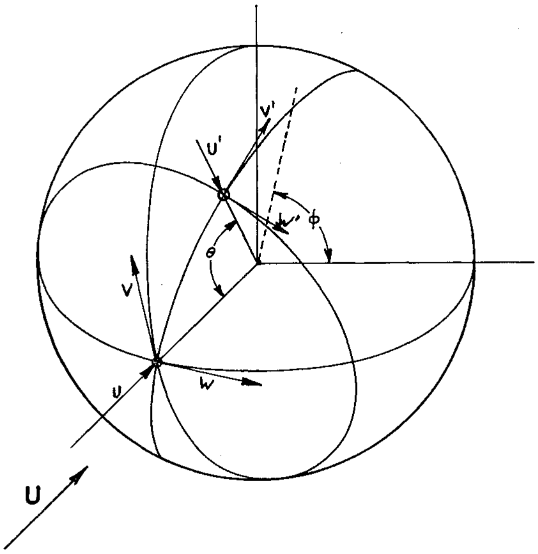


**Fig. 1 Spherical coordinate system.**

where the distribution function $f$ is taken from Grad[3]:

$$f = f^0 \left[1 + \frac{1}{2pRT}\left\{p_{xx}(u - U)^2 + p_{yy}v^2 + p_{zz}w^2 - 2\tau_{xy}(u - U)v - 2\tau_{xz}(u - U)w - 2\tau_{yz}vw - 2\left(1 - \frac{(u - U)^2 + v^2 + w^2}{5RT}\right)[q_x(u - U) + g_y v + q_z w]\right\}\right] \quad (5)$$

where $f^0$ is the Maxwellian distribution function given by

$$f^0 = \frac{\rho}{m(2\pi RT)^{3/2}} \exp\left(- \frac{(u - U)^2 + v^2 + w^2}{2RT}\right) \quad (6)$$

The symbols $\rho, p, R, T, U, p_{xx}, p_{yy}, p_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yz}, q_x, q_y,$ and $q_z$ refer to the gas density, pressure, gas constant, temperature, velocity, the normal and tangential stresses, and heat flux components. The symbols $m, u, v,$ and $w$ denote the molecular mass and velocity components. The primed and unprimed coordinate system are shown in Fig. 1. The two systems are related thus:

$$\begin{aligned} u &= u' \cos\theta + v' \sin\theta \\ v &= -u' \sin\phi \sin\theta + v' \sin\phi \cos\theta - w' \cos\phi \\ w &= -u' \cos\phi \sin\theta + v' \cos\phi \cos\theta - w' \sin\phi \end{aligned} \quad (7)$$

To integrate the energy flux equation, Eq. (3), the thermal velocities are introduced:

$$\begin{aligned} c_1 &= u' - U \cos\theta \\ c_2 &= v' - U \sin\theta \\ c_3 &= w' \end{aligned} \quad (8)$$

Substituting these in Eq. (4) yields

$$\frac{dE_i}{dA} = \frac{m}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-v\cos\theta}^{\infty} (c_1^2 + c_1^2 + c_3^2 + 2c_1U \cos\theta + 2c_2U \sin\theta + U^2)(c_1 + U \cos\theta) f dc_1 dc_2 dc_3 \ldots \quad (9)$$

The energy flux per unit area of diffusely reflected molecules is given by

$$\frac{dE_w}{dA} = 2mRT_w \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} u' f du' dv' dw' \quad (10)$$

Substituting the new coordinates into Eq. (10) yields

$$\frac{dE_w}{dA} = 2mRT_w \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-U\cos\theta}^{\infty} (c_1 + U \cos\theta) f dc_1 dc_2 dc_3 \quad (11)$$

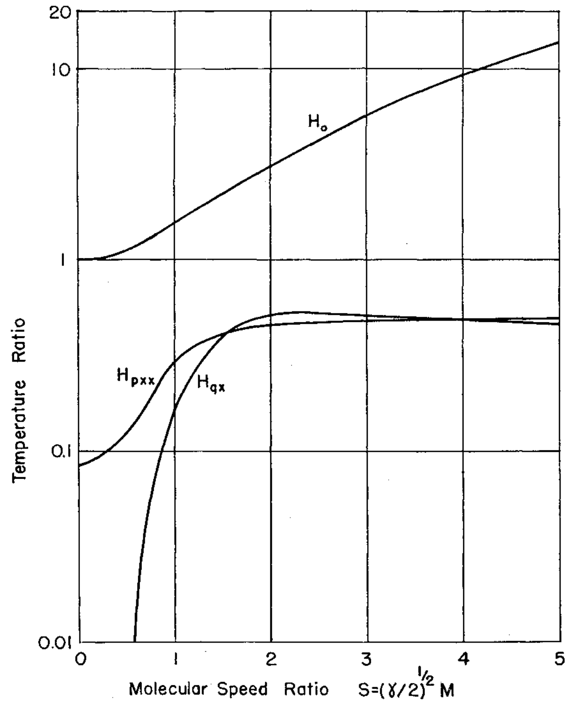


**Fig. 2 Partial temperature ratios for nonuniform free molecule flow past a sphere.**

Performing the indicated integration over the velocity space and then integrating over the sphere area

$$dA = r^2 \sin\theta d\theta d\varphi$$

one has‡

$$\begin{aligned} \frac{1}{\alpha} Q = \frac{\pi r^2 \rho URT}{S^4} \Bigg\{ &\left[\operatorname{erf}(S)\left(S^4 + 3S^2 + \frac{3}{4}\right) + \frac{S}{\pi^{1/2}} \exp(-S^2)\left(S^2 + \frac{5}{2}\right)\right] S^2 + \frac{p_{xx}}{4p}\Bigg[\operatorname{erf}(S) \times \\ &\left(2S^6 + 12S^4 + \frac{3}{2}S^2 - \frac{1}{2}\right) + \frac{S}{\pi^{1/2}} \exp(-S^2) \times \left(2S^4 + \frac{1}{S^2} + 1\right)\Bigg] + \frac{p_{yy}}{4p}\Bigg[\operatorname{erf}(S)\Bigg(2S^6 + 9S^4 + \\ &\frac{9}{2}S^2 - \frac{3}{4}\Bigg) + \frac{S}{\pi^{1/2}} \exp(-S^2)\left(2S^4 + 8S^2 + \frac{3}{2}\right)\Bigg] + \\ &\frac{p_{zz}}{4p}\left[\operatorname{erf}(S)\left(2S^6 + 9S^4 + \frac{9}{2}S^2 - \frac{3}{4}\right) + \frac{S}{\pi^{1/2}} \exp(-S^2)\left(2S^4 + 8S^2 + \frac{3}{2}\right)\right] + \\ &\frac{q_x S^2}{5pU}\left[\operatorname{erf}(S)\left(S^2 + \frac{3}{2}\right) - \frac{S}{\pi^{1/2}} \exp(-S^2)(5S^2 + 3)\right]\Bigg\} - \\ &\frac{r^2 \rho URT_w}{S^4}\Bigg\{\left[\operatorname{erf}(S)(2S^2 + 1) + \frac{2S}{\pi^{1/2}} \exp(-S^2)\right] S^2 + \\ &\frac{p_{xx}}{4p}\left[\operatorname{erf}(S)(4S^4 + 2S^2 + 2) + \frac{4S}{\pi^{1/2}} \exp(-S^2)(S^2 - 1)\right] + \\ &\frac{p_{yy}}{4p}\left[\operatorname{erf}(S)(4S^4 + 4S^2 + 4) - \frac{8S}{\pi^{1/2}} \exp(-S^2)\right] + \\ &\frac{p_{zz}}{4p}\left[\operatorname{erf}(S)(4S^4 + 4S^2 + 4) - \frac{8S}{\pi^{1/2}} \exp(-S^2)\right] - \\ &\frac{q_x S^2}{5pU}\left[\frac{\operatorname{erf}(S)}{4} - \frac{S}{2\pi^{1/2}} \exp(-S^2)\right]\Bigg\} \end{aligned} \quad (12)$$

‡ Note that the terms $\tau_{xy}, \tau_{xz}, \tau_{yz}, q_y,$ and $q_z$ drop out after integration (because of symmetry or over the area integration).

where erf$(S)$ is the error function defined by

$$\operatorname{erf}(S) = \frac{2}{\pi^{1/2}} \int_0^S e^{-x^2}\, dx \qquad (13)$$

and

$$S = U/(2RT)^{1/2} = (\gamma/2)^{1/2}M$$

From the definition of the stress tensor, one obtains for the diagonal terms

$$P_{ii} = 3p$$

also

$$p_{ij} = P_{ij} - p\delta_{ij} \qquad (14)$$

and hence

$$p_{ij} = 0 \text{ (i.e., } p_{xx} + p_{yy} + p_{zz} = 0)$$

From the foregoing relationship, if one substitutes $-(p_{xx} + p_{yy})$ for $p_{zz}$, both $p_{zz}$ and $p_{yy}$ terms drop out.

One should note that these results are confined to the case of a monatomic gas, since there still exists a question as to the nature of the distribution function analogous to that given by Eq. (5) but applying to a diatomic gas. For adiabatic wall conditions $Q = 0$. If $T_w$ is denoted by $T_{aw}$, the resulting equilibrium sphere temperature can be written as

$$\frac{T_{aw}}{T} = H_0 + \frac{p_{xx}}{p} H_{pxx} + \frac{q_x}{pU} H_{qx}$$

where

$$H_0 = \frac{\operatorname{erf}(S)(S^4 + 3S^2 + \frac{3}{4}) + [S\exp(-S^2)/\pi^{1/2}]\,(S^2 + \frac{5}{2})}{\operatorname{erf}(S)(2S^2 + 1) + (2S/\pi^{1/2})e^{-S^2}}$$

$$H_{pxx} = \frac{\operatorname{erf}(S)(3S^4 - 3S^2 + \frac{1}{4}) + [S\exp(-S^2)/\pi^{1/2}](3S^2 - \frac{1}{2}) + [\operatorname{erf}(S)(2S^2 + 2) - 4S\exp(-S^2)/\pi^{1/2}(S^2 + 1)]H_0}{4[\operatorname{erf}(S)(2S^2 + 1) + (2Se^{-S^2}/\pi^{1/2})]S^2} \qquad (15)$$

$$H_{qx} = \frac{4\{\operatorname{erf}(S)(S^2 + \frac{3}{2}) - [S\exp(-S^2)/\pi^{1/2}](5S^2 + 3) + [\frac{1}{4}\operatorname{erf}(S) - S\exp(-S^2)/2\pi^{1/2}]H_0\}}{5[\operatorname{erf}(S)(2S^2 + 1) + (2S/\pi^{1/2})e^{-S^2}]}$$

These partial equilibrium temperature ratios are plotted in Fig. 2, where they indicate that for $S \gg 1$ the effects due to viscous stresses and heat flux become relatively unimportant.

For the heat transfer case, one defines a Stanton number $St$ for sphere:

$$St = Nu/RePr = [\alpha(\gamma - 1)/\gamma]B(S)$$

where

$$B(S) = \left[\operatorname{erf}(S)(2S^2 + 1) + \frac{2S}{\pi^{1/2}}\exp(-S^2)\right]\frac{1}{S^2} + \frac{p_{xx}}{4p}\left[\operatorname{erf}(S)(2S^2 + 2) - \frac{4S}{\pi^{1/2}}\exp(-S^2)(S^2 + 1)\right] - \frac{q_xS^2}{5pU}\left[\frac{1}{4}\operatorname{erf}(S) - \frac{S}{\pi^{1/2}}\exp(-S^2)\right] \qquad (16)$$

$$\frac{1}{\alpha} St = St_0 + \frac{p_{xx}}{p} St p_{xx} - \frac{q_x}{pU} St_{qx} \qquad (17)$$

where

$$\frac{1}{\alpha} St_0 = \frac{2}{5}\left[\operatorname{erf}(S)(2S^2 + 1) + \frac{2S}{\pi^{1/2}}\exp(-S^2)\right]\cdot\frac{1}{S^2}$$

$$\frac{1}{\alpha} St_{pxx} = \frac{1}{10S^4}\left[\operatorname{erf}(S)(2S^2 + 2) - \frac{4S}{\pi^{1/2}}\exp(-S^2)(S^2 + 1)\right]$$

$$\frac{1}{\alpha} Stq_x = \left[\frac{\operatorname{erf}(S)}{4} - \frac{S}{2\pi^{1/2}}\exp(-S^2)\right]\frac{2}{25S^2} \qquad (18)$$

These partial Stanton numbers are plotted in Fig. 3.

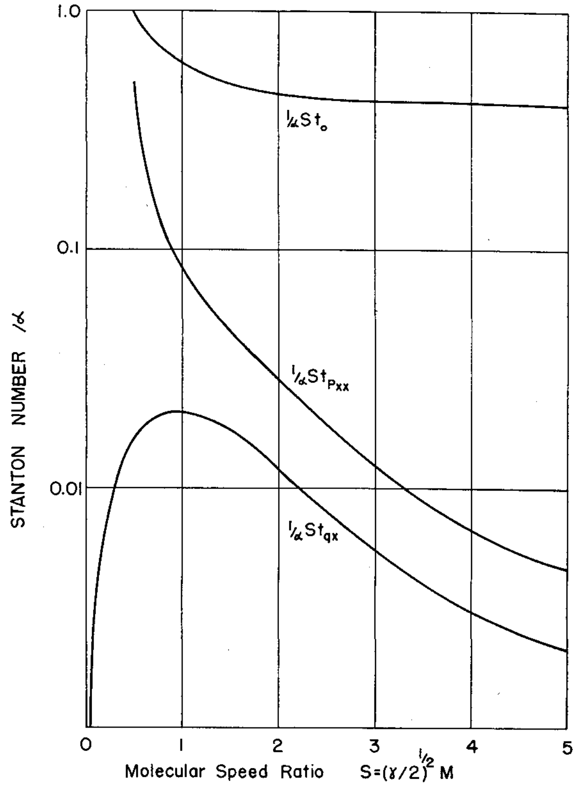


**Fig. 3 Partial Stanton numbers for nonuniform free molecule flow past a sphere.**


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# Acoustic Probe for Hypersonic Air Data Sensing

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## Nomenclature

$u$ = gas velocity
$\gamma$ = isentropic expansion exponent
$P$ = pressure
$\rho$ = density
$a$ = sonic velocity

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